Symmetries of Tensor Products

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ABSTRACT

We give sufficient conditions for the description of the isometry group of a tensor product in terms of the isometry groups of the factors. A slightly more general theory involving tensor products and saturated linear groups is developed for this purpose.

0. INTRODUCTION

Let U and V be vector spaces, and endow the tensor product

$$W = U \otimes V$$

with the canonical norm (see Section 7 below). Then any linear automorphism c of W of form

$$c = a \otimes b, \tag{1}$$

where a and b are isometries of U and V respectively, is an isometry of W. Moreover, if U and V are isometrically isomorphic, then any automorphism c

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of form

$$c(x\otimes y)=(ay)\otimes(bx), \tag{2}$$

where $a: V\to U$ and $b: U\to V$ are isometries, is also an isometry of W.

There are examples (see Section 4 below) which show that these need not be the only examples. Theorem 8.7 below says that every isometry of W has one of the forms (1) or (2) in case the dual norms on U^* and V^* are smooth, and Theorem 8.6 below gives a sufficient condition that every c in the identity component of the group of isometries of W have the form (1).

These theorems are derived from a slightly more general theory (developed in Sections 2–5) concerning tensor products and "saturated" linear groups, i.e. groups which are the (setwise) stabilizer of some set. The reason that these groups arise naturally in this context is the theorem of Robbin [7] (see also [4]) that a linear group is the isometry group of some norm iff it is compact, is saturated, and contains all unit norm scalar matrices.

We now list briefly the content of the paper. In Section 1 we introduce basic notations and concepts which are used in the sequel. In Section 2 we define the wreath tensor product of groups, which is naturally associated with the groups G_{ij}—the subgroup of linear isomorphisms from the space W_i to W_j which preserve a given structure. We then state our "guiding principle" for the wreath tensor product. The well-known results of Marcus and Moyls [5] and others is one of the cases where our principle applies. Section 3 deals with stable groups $\mathrm{GL}(V,Y)$ which consist of all linear isomorphisms $a: V\to V$ such that $aY=Y$. This concept is a natural generalization of the concept of the saturation. Theorems 3.1–3.3 describe three distinct situations where the "guiding principle" applies. The proof of these theorems is given in Section 5. Section 4 is devoted to simple examples to illustrate Theorems 3.1–3.3 and to show that some conditions in these theorems cannot be dropped. In Section 6 we introduce more notation related to the normed spaces which is needed in the next section. Section 7 is devoted to the discussion of the canonical norms on the tensor product of normed spaces.

In Section 8 we give our main results. Theorem 8.6 essentially claims that the identity component of $W=U\otimes V$ has the form (1). Theorem 8.7 says that every isometry of W is of the form (1) if the norms on U and V (or on U^* and V^*) are smooth. The last section is devoted to remarks and conjectures.

1. SOME NOTATION

Throughout, $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. All vector spaces will be finite dimensional over $\mathbb{F}$.

For vector spaces $U, V, W_1, \dots, W_n$ denote by $L^n(W_1, \dots, W_n; V)$ the space of all multilinear maps from $W_1 \times \cdots \times W_n$ to V; by $L(U,V)$ the space of linear maps from U to V; by $L_{\mathrm{iso}}(U,V)$ the (possibly empty) set of all linear isomorphisms from U to V; by $\mathrm{GL}(V)$ the general linear group of V; and by $\mathrm{gl}(V)$ the Lie algebra of $\mathrm{GL}(V)$. Thus:

$$L(U,V) = L^1(U,V),$$

$$V^* = L(V, \mathbb{F}),$$

$$\mathrm{GL}(V) = L_{\mathrm{iso}}(V,V),$$

$$\mathrm{gl}(V) = L(V,V).$$

We shall denote by W the tensor product

$$W = W_1 \otimes \cdots \otimes W_n,$$

by $\mu \in L^n(W_1, \dots, W_n; W)$ the canonical map

$$\mu(x_1, \dots, x_n) = x_1 \otimes \cdots \otimes x_n,$$

and by M the image of μ:

$$M = \{x_1 \otimes \cdots \otimes x_n : x_j \in W_j\}.$$

Elements of M are called *decomposable* (or rank one by some authors).

For every vector space V the map μ induces a vector-space isomorphism:

$$\begin{aligned} &\mu^* : L(W,V) \to L^n(W_1, \dots, W_n; V), \\ &(\mu^* A)(x_1, \dots, x_n) = A(x_1 \otimes \cdots \otimes x_n). \end{aligned} \tag{1.1}$$

This fact is called the "universal mapping property" and characterizes uniquely the tensor product W (or rather the map μ.)

By elimination theory (see e.g. [8, p. 104]) M is an algebraic variety. This can be seen directly as follows:

PROPOSITION 1.2. *A point $w \in W$ lies in M iff it satisfies all the quadratic equations*

$$\langle \xi, w \rangle \langle \eta, w \rangle = \langle \xi', w \rangle \langle \eta', w \rangle,$$

where $\xi, \eta, \xi', \eta' \in W^*$ range over all quadruples of form

$$\xi = \xi_1 \otimes \cdots \otimes \xi_n, \qquad \xi' = \xi'_1 \otimes \cdots \otimes \xi'_n,$$

$$\eta = \eta_1 \otimes \cdots \otimes \eta_n, \qquad \eta' = \eta'_1 \otimes \cdots \otimes \eta'_n,$$

where $\xi_j, \eta_j \in W_j^*$ and $\{\xi_j, \eta_j\} = \{\xi'_j, \eta'_j\}$ for $j = 1, \ldots, n$ (here $\{\xi, \eta\}$ is the set consisting of the elements ξ and η).

2. THE WREATH TENSOR PRODUCT

Imagine that each of the spaces W_j has some structure, and denote by G_{ij} the set of linear isomorphisms from W_i to W_j which preserve that structure. (We allow G_{ij} to be empty for $i \neq j$.) More specifically, assume given

$$G_{ij} \subset L_{\text{iso}}(W_i, W_j)$$

satisfying the following axioms:

$$a \in G_{ij}, \quad b \in G_{jk} \quad \Rightarrow \quad ba \in G_{ik}; \tag{2.1}$$

$$a \in G_{ij} \quad \Rightarrow \quad a^{-1} \in G_{ji}; \tag{2.2}$$

$$e \in G_{ii} \tag{2.3}$$

(where e denotes the identity map). Note that in particular

$$G_{ii} \subset \mathrm{GL}(W_i)$$

is a subgroup.

Given such a system, we define a new group

$$G \subset \mathrm{GL}(W)$$

called the *wreath tensor product*; it consists of all transformations $a \in \mathrm{GL}(W)$ of the form

$$a(x_1 \otimes \cdots \otimes x_n) = a_1 x_{\sigma(1)} \otimes \cdots \otimes a_n x_{\sigma(n)} \tag{2.4}$$

for $x_j \in W_j$, where σ is a permutation of $\{1,\dots,n\}$ and $a_j \in G_{\sigma(j)j}$. The idea of this definition is the following

GUIDING PRINCIPLE 2.5. *Whenever a structure is defined in W from the given structures in $W_1,\dots,W_n$, the group G should be a subgroup of the automorphism group of that structure.*

We denote by G_0 the subgroup of G consisting of those a given by (2.4) for which σ is the identity permutation. Note the natural surjective homomorphism

$$\begin{gathered} G_{11} \times \cdots \times G_{nn} \to G_0 \\ (a_1,\dots,a_n) \mapsto a_1 \otimes \cdots \otimes a_n. \end{gathered} \tag{2.6}$$

The kernel of the homomorphism (2.6) consists of n-tuples of scalars whose product is the identity. The group G_0 is a normal subgroup of G of finite index; the quotient group consists of all permutations σ for which $G_{\sigma(j)j} \neq \varnothing$ for $j = 1,\dots,n$.

We shall denote the Lie algebra of G_{jj} by $\mathcal{G}_j$ and the common Lie algebra of G and G_0 by

$$\mathcal{G} = \{A \in \mathrm{gl}(W) : \exp(tA) \in G \ \forall t \in \mathbb{R}\}.$$

The homomorphism of Lie algebras

$$\mathcal{G}_1 \times \cdots \times \mathcal{G}_n \to \mathcal{G}$$

induced by (2.6) is an isomorphism when each $\mathcal{G}_j$ consists of matrices of trace zero.

The following well-known theorem (see [5], [6], and [9]) illustrates the kind of result we want to prove.

THEOREM 2.7. *Suppose*

$$G_{ij} = L_{\mathrm{iso}}(W_i, W_j).$$

Then

$$G = \mathrm{GL}(W, M),$$

where $M \subset W$ is the manifold of decomposable elements and $\mathrm{GL}(W, M)$ *is by* *definition the group of "decomposable preservers"*:

$$\mathrm{GL}(W,M)=\{a\in\mathrm{GL}(W): a(M)=M\}.$$

Note that M is "ruled": through each point $x=x_1\otimes\cdots\otimes x_n$ there pass n linear subspaces:

$$R_j(x)=x_1\otimes\cdots\otimes x_{j-1}\otimes W_j\otimes x_{j+1}\otimes\cdots\otimes x_n.$$

Moreover for each $j=1,\dots,n$ we have

$$M=\bigcup_x R_j(x),$$

$$R_j(x)\cap R_j(x')\neq\{0\}\quad\Leftrightarrow\quad R_j(x)=R_j(x');$$

and for $i\neq j$

$$R_i(x)\cap R_j(x)=\mathbb{F}x.$$

Any element $a\in\mathrm{GL}(W,M)$ preserves this ruled structure in the sense that

$$aR_{\sigma(j)}(x)=R_j(ax)$$

for $j=1,\dots,n$, $x\in M$. This can be seen directly (without appealing to Theorem 2.7) as follows: for $x\in M\setminus\{0\}$ let $T_xM\subset W$ denote the tangent space to M at x:

$$T_xM=\{\dot c(0)\,|\,c:\mathbb{R}\to M,\ c(0)=x\}.$$

Then

$$M\cap T_xM=\bigcup_j R_j(x)$$

and for $a\in\mathrm{GL}(W,M)$,

$$a(T_xM)=T_{ax}M.$$

3. THE SETWISE STABILIZER GL(V, Y)

For any subset $Y \subset V$ we denote by GL(V, Y) the setwise stabilizer of Y in GL(V):

$$\mathrm{GL}(V, Y) = \{a \in \mathrm{GL}(V) : a(Y) = Y\}.$$

Subgroups of GL(V) of form GL(V, Y) are called *saturated* (see [4], [7]). One easily checks that a subgroup $G \subset \mathrm{GL}(V)$ is saturated iff it is defined by its orbits, i.e., iff $a \in G$ whenever $a \in \mathrm{GL}(V)$ and $ax \in Gx$ for all $x \in V$. We denote by gl(V, Y) the Lie algebra of GL(V, Y):

$$\mathrm{gl}(V, Y) = \{A \in \mathrm{gl}(V) : \exp(tA)(Y) = Y \ \forall t \in \mathbb{R}\}.$$

Call a subset Y of $V \setminus \{0\}$ *homogeneous* iff it is invariant under multiplication by nonzero scalars:

$$y \in Y, \quad \lambda \in \mathbb{F} \setminus \{0\} \quad \Rightarrow \quad \lambda y \in Y;$$

circled iff it is invariant under multiplication by scalars of absolute value one:

$$y \in Y, \quad \lambda \in \mathbb{F}, \quad |\lambda| = 1 \quad \Rightarrow \quad \lambda y \in Y;$$

antiradial iff it intersects each ray in at most one point:

$$y \in Y, \quad t > 0, \quad ty \in Y \quad \Rightarrow \quad t = 1.$$

Call two subsets Y_1 and Y_2 of $V \setminus \{0\}$ *linearly disjoint* iff

$$\mathrm{span}(Y_1) \cap \mathrm{span}(Y_2) = \{0\},$$

and call a subset Y of $V \setminus \{0\}$ *linearly prime* iff it is not the union of two nonempty linearly disjoint subsets. Clearly every subset Y of $V \setminus \{0\}$ has a unique decomposition:

$$Y = Y_1 \cup \cdots \cup Y_k$$

where $Y_1, \ldots, Y_k$ are nonempty and linearly prime, and Y_i and $Y \setminus Y_i$ are linearly disjoint. When $V=\operatorname{span}(Y)$ we have the obvious inclusion of groups:

$$\mathrm{GL}(V_1,Y_1)\times\cdots\times\mathrm{GL}(V_k,Y_k)\subset\mathrm{GL}(V,Y)$$

and isomorphism of Lie algebras:

$$\mathrm{gl}(V_1,Y_1)\times\cdots\times\mathrm{gl}(V_k,Y_k)=\mathrm{gl}(V,Y),$$

where $V_j=\operatorname{span}(Y_j)$ for $j=1,\dots,k$, so that

$$V=V_1\oplus\cdots\oplus V_k.$$

Now fix $Z_j\subset W_j\setminus\{0\}$ $(j=1,\dots,n)$ and assume

$$W_j=\operatorname{span}(Z_j).$$

Let $Z\subset W$ denote the image of $Z_1\times\cdots\times Z_n$ under the canonical map $\mu\in L^n(W_1,\dots,W_n;W)$:

$$Z=\{z_1\otimes\cdots\otimes z_n : z_j\in Z_j\}$$

For $i,j=1,\dots,n$ set

$$G_{ij}=\{a\in L_{\mathrm{iso}}(W_i,W_j) : a(Z_i)=Z_j\},$$

so that

$$G_{jj}=\mathrm{GL}(W_j,Z_j),$$

$$\mathcal{G}_j=\mathrm{gl}(W_j,Z_j).$$

The system $\{G_{ij}\}$ satisfies the axioms (2.1)–(2.3), so we may form the group G and Lie algebra $\mathcal{G}$. We have (by the guiding principle) the obvious inclusions:

$$G\subset\mathrm{GL}(W,Z),$$

$$\mathcal{G}\subset\mathrm{gl}(W,Z);$$

we wish to investigate the extent to which these inclusions are equalities. (Note that Theorem 2.7 asserts equality in case $Z_j=W_j\setminus\{0\}$.)

THEOREM 3.1. *Suppose each Z_i is linearly prime. Then Z is linearly prime and*

$$\mathrm{gl}(W, Z) \subset \mathrm{gl}(W, M).$$

THEOREM 3.2. *Suppose each Z_i is linearly prime and homogeneous. Then Z is linearly prime and homogeneous, and*

$$\mathcal{G} = \mathrm{gl}(W, Z).$$

THEOREM 3.3. *Suppose each Z_i is linearly prime, circled, and antiradial. Then Z is linearly prime, circled, and antiradial, and*

$$\mathcal{G} = \mathrm{gl}(W, Z).$$

4. EXAMPLES

Before proving these theorems, we give some examples to show that they cannot be strengthened.

EXAMPLE 4.1. Let $W_1 = V_1 \otimes U$, $W_2 = V_2 \otimes U$, where $\dim(V_1) > 1$ and $\dim(U) > 1$. Choose $Y_i \subset V_i \setminus \{0\}$, $X \subset U \setminus \{0\}$ which span the corresponding spaces and are linearly prime, and let $Z_i = \{y_i \otimes x : y_i \in Y_i,\ x \in X\}$. Then

$$\mathrm{GL}(W, Z) \not\subset \mathrm{GL}(W, M),$$

as the transformation

$$y_1 \otimes x \otimes y_2 \otimes x' \to y_1 \otimes x' \otimes y_2 \otimes x$$

lies in $\mathrm{GL}(W, Z)$ but not in $\mathrm{GL}(W, M)$. By Theorem 3.1 each Z_i is linearly prime; hence the conclusion of Theorem 3.1 cannot be strengthened to the analogous assertion about groups. One can arrange that the Z_i satisfy the hypotheses of Theorem 3.2 (or Theorem 3.3) by choosing Y_i and X to satisfy them; but

$$G \subset \mathrm{GL}(W, M)$$

by definition, so that we have examples where

$$G \neq \mathrm{GL}(W, Z),$$

showing that the conclusion of Theorem 3.2 (or Theorem 3.3) also need not hold on the level of groups.

EXAMPLE 4.2. Take $n=2$, $\dim(W_1)=\dim(W_2)=2$, and let $\{e_{j1}, e_{j2}\}$ be a basis for W_j $(j=1,2)$. Let

$$Z_j = (\mathbb{F}e_{j1} \cup \mathbb{F}e_{j2}) \setminus \{0\},$$

so that Z_j is homogeneous but not linearly prime. Define $a \in \mathrm{GL}(W)$ by

$$a(e_{1r} \otimes e_{2s}) = \lambda_{rs} e_{1r} \otimes e_{2s}.$$

Then $a \in \mathrm{GL}(W, Z)$ but $a \notin \mathrm{GL}(W, M)$ if $\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}$ (compare Proposition 1.1). Note that a may be chosen arbitrarily near the identity, so that Theorem 3.1 does not hold without the hypothesis that the Z_j are linearly prime even when the Z_j are homogeneous.

EXAMPLE 4.3. This is the same as Example 4.2, but take $\mathbb{F} = \mathbb{C}$ and

$$Z_j = S^1 e_{j1} \cup S^1 e_{j2},$$

where $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and choose $\lambda_{rs} \in S^1$. Again the conclusion of Theorem 3.1 fails although the Z_j are circled and antiradial.

EXAMPLE 4.4. Take $n=2$, $\mathbb{F}=\mathbb{R}$:

$$W_j = V_{j1} \oplus V_{j2},$$

where V_{jr} is a Hilbert space and $\dim(V_{jr}) > 1$, and let

$$Z_j = Y_{j1} \cup Y_{j2},$$

where Y_{jr} is the unit sphere in V_{jr}. Consider a transformation $a \in \mathrm{GL}(W)$ of form

$$a(y_{1r} \otimes y_{2s}) = (b_{1r} y_{1r}) \otimes (b_{2s} y_{2s}),$$

where $y_{jr} \in V_{jr}$ and $b_{jr} \in \mathrm{GL}(V_{jr})$. If each b_{jr} is orthogonal, we clearly have that $a \in \mathrm{GL}(W, Z)$, but we will not have $a \in \mathrm{GL}(W, M)$ in general, even when a

is very close to the identity. Again the conclusion of Theorem 3.1 fails although $\mathbb{F} = \mathbb{R}$ and each Z_j is antiradial.

EXAMPLE 4.5. Take $n = 2$, $\dim(W_j) > 1$, and suppose W_1 is a Hilbert space. Let $Z_1 =$ unit sphere of W_1 and $Z_2 = W_2 \setminus \{0\}$. Then $Z = M$. This example shows that $\mathrm{gl}(W, Z)$ can be much larger than $\mathcal{G}$ even when the Z_j are linearly prime. In other words, the common conclusion of Theorems 3.2 and 3.3 does not follow from the hypothesis of Theorem 3.1.

5. PROOFS OF THEOREMS 3.1, 3.2, 3.3

First note that Theorem 3.1 follows from Theorem 3.2. Indeed, if we set $\tilde{Z}_j = (\mathbb{F} \setminus \{0\}) \cdot Z_j$, then $\tilde{Z}_j$ is homogeneous and $\mathrm{GL}(W, Z) \subset \mathrm{GL}(W, \tilde{Z})$. Next note that in Theorems 3.2 and 3.3 it suffices to consider the case $n = 2$ by induction. Hence without loss of generality we assume $n = 2$; to simplify the notation we write

$$U = W_1, \qquad V = W_2,$$

$$X = Z_1, \qquad Y = Z_2,$$

so that

$$W = U \otimes V;$$

$$Z = \{x \otimes y : x \in X,\, y \in Y\}.$$

We first show that Z is linearly prime. To this end suppose

$$Z = Z^1 \cup Z^2,$$

where Z^1 and Z^2 are linearly disjoint and nonempty; we shall derive a contradiction. For $y \in Y$ and $r = 1,2$ let

$$X^r(y) := \{x \in X : x \otimes y \in Z^r\}.$$

Then $X^1(y)$ and $X^2(y)$ are linearly disjoint (or empty) and

$$X = X^1(y) \cup X^2(y),$$

so either $X^1(y) = X$ or $X^2(y) = X$. Hence let

$$Y^r := \{y \in Y : X^r(y) = X\}.$$

Then $Y = Y^1 \cup Y^2$. It is enough to show that Y^1 and Y^2 are linearly disjoint, for then one of them, say Y^1, must be empty, whence Z^1 is empty, the desired contradiction. But if $v \in \operatorname{span}(Y^1) \cap \operatorname{span}(Y^2)$, then for $x \in X$ we have $x \otimes v \in \operatorname{span}(x \otimes Y^1) \cap \operatorname{span}(x \otimes Y^2)$. Since $x \otimes Y^r \subset Z^r$ and $\operatorname{span}(Z^1) \cap \operatorname{span}(Z^2) = \{0\}$, it follows that $v = 0$ as required.

We now continue with the proof of Theorem 3.1. Choose $C \in \mathrm{gl}(W, Z)$, and set

$$c = \exp(tC)$$

with $t \in \mathbb{R}$. We must show that $c(M) = M$. First we show the following:

Claim. *For each $y \in Y$ there exists $y' \in Y$ with*

$$c(U \otimes y) = U \otimes y'.$$

Fix y and choose a basis $e_1, \ldots, e_m$ of U with $e_r \in X$ $(r = 1, \ldots, m)$. As $c(Z) = Z$, we may write

$$c(e_r \otimes y) = e_r' \otimes y_r'$$

for $r = 1, \ldots, m$. Assume temporarily that c is close to the identity; then $e_1', \ldots, e_m'$ form a basis, and (rescaling if necessary) we may take y_r' close to y. (We do not require $y_r' \in Y$.)

Partition the set $\{1, \ldots, m\}$ into equivalence classes according to the equivalence relation $\mathbb{F} y_r' = \mathbb{F} y_s'$. Thus for each equivalence class R we may set y_R' equal to one of the y_r' with $r \in R$ and obtain

$$c(e_r \otimes y) \in U \otimes y_R'$$

for $r \in R$. Rescaling the e_r' if necessary, we may as well assume

$$c(e_r \otimes y) = e_r' \otimes y_R' \tag{5.1}$$

for $r \in R$.

The claim asserts that there is only one equivalence class R. Since X is linearly prime, we may prove this by showing that

$$X \subset \bigcup_R \operatorname{span}\{e_r : r \in R\}.$$

Hence choose $x \in X$. Using the basis $e_1, \dots, e_m$, write x in the form

$$x = \sum x_R,$$

where

$$x_R \in \operatorname{span}\{e_r : r \in R\}.$$

As $x \otimes y \in Z = c(Z)$, we have

$$c(x \otimes y) = x' \otimes y'$$

for some $y' \in Y$. But by linearity and (5.1),

$$c(x \otimes y) = \sum c(x_R \otimes y) = \sum x'_R \otimes y'_R,$$

where

$$x'_R \in \operatorname{span}\{e'_r : r \in R\}.$$

(Note that $x'_R = 0 \Leftrightarrow x_R = 0$.) Hence

$$\sum x'_R \otimes y'_R = x' \otimes y'. \tag{5.2}$$

For each nonzero x_R we can find (since $e'_1, \dots, e'_m$ form a basis) a functional ξ with $\langle \xi, x'_R \rangle = 1$ and $\langle \xi, x'_S \rangle = 0$ for $S \neq R$. Applying ξ to (5.2) yields

$$y'_R = \langle \xi, x' \rangle y'.$$

Hence by the definition of the equivalence relation, at most one x_R is nonzero; i.e., $x = x_R$, as was to be shown.

Now we must remove the assumption that c is close to the identity. We have shown that for each y there is an $\varepsilon > 0$ such that the claim holds for $c = \exp(tC)$ with $|t| < \varepsilon$. By compactness (in projective space) ε may be chosen independent of y. Hence by iterating c the claim holds for all t as required.

By exactly the same argument we also have the following:

Claim. *For each $x \in X$ there exists $x' \in X$ with*

$$c(x \otimes V) = x' \otimes V.$$

Now choose a basis $e_1,\dots,e_m \in X$ of U and a basis $f_1,\dots,f_n \in Y$ of V. By the claims, choose $e_1',\dots,e_m' \in X$, $f_1',\dots,f_n' \in Y$ such that

$$c(e_r \otimes V) = e_r' \otimes V,$$

$$c(U \otimes f_s) = U \otimes f_s'$$

for $r = 1,\dots,m$, $s = 1,\dots,n$. Then

$$\begin{aligned} c(e_r \otimes f_s) &\in c(e_r \otimes V \cap U \otimes f_s) \\ &= e_r' \otimes V \cap U \otimes f_s' \\ &= \mathbb{F} \cdot e_r' \otimes f_s', \end{aligned}$$

so there exist scalars $c_{rs} \in \mathbb{F} \setminus \{0\}$ with

$$c(e_r \otimes f_s) = c_{rs} e_r' \otimes f_s'.$$

We must find scalars a_r, b_s with

$$c_{rs} = a_r b_s$$

for then $c = a \otimes b$, where

$$ae_r = a_r e_r',$$

$$bf_s = b_s f_s'.$$

Notice that Examples 4.2–4.4 satisfy the conclusions of the claims, so we must again use the assumption that X and Y are linearly prime. Such scalars a_r and b_s exist iff for all indices r, r', s, s' we have

$$(*) \qquad c_{rs} c_{r's'} = c_{rs'} c_{r's}.$$

(Indeed, the existence of the scalars is equivalent to the condition that c_{rs} is a rank one matrix, which is equivalent to the condition that each two by two minor vanishes.)

Suppose

$$x = \sum \alpha_i e_i \in X,$$

$$y = \sum \beta_j e_j \in Y.$$

Then

$$c(x\otimes y)=\sum c_{ij}\alpha_i\beta_j e_i'\otimes f_j'=x'\otimes y',$$

so by Proposition 1.2,

$$c_{rs}\alpha_r\beta_s c_{r's'}\alpha_{r'}\beta_{s'}=c_{rs'}\alpha_r\beta_{s'}c_{r's}\alpha_{r'}\beta_s,$$

so $(*)$ holds when $\alpha_r, \alpha_{r'}, \beta_s, \beta_{s'}\neq 0$. Now for any proper subset of indices $I\subset\{1,\dots,m\}$ we have

$$X\not\subset \operatorname{span}\{e_i : i\in I\}\cup\operatorname{span}\{e_i : i\notin I\},$$

so we can find x with $\alpha_r\neq 0$ for some $r\in I$ and $\alpha_{r'}\neq 0$ for some $r'\notin I$. Let $I_1=\{1\}$, and (if I_i is defined for $i<n$) read I_i for I in the last sentence and set $I_{i+1}=I_i\cup\{r'\}$. Rename the indices so $I_i=\{1,\dots,i\}$. Then for each $r'=1,\dots,m$ there exist $r<r'$ and $x\in X$ such that $\alpha_r, \alpha_{r'}\neq 0$; similarly for Y. Hence for each $r'=1,\dots,m$ and each $s'=1,\dots,n$ there exist $r<r'$ and $s<s'$ such that $(*)$ holds. If (for fixed r, r') $(*)$ holds for $(s,s')=(j,j')$ and also for $(s,s')=(j',j'')$, then it holds for $(s,s')=(j,j'')$ as $c_{rs}\neq 0$. Similarly, if for $(r,r')=(i,i')$ and $(r,r')=(i',i'')$ $(*)$ holds for *all* (s,s'), then it does also for $(r,r')=(i,i'')$. Hence by induction $(*)$ holds identically, as required.

This proves Theorem 3.1. Note that we have also given a proof of Theorem 2.7, since we found a and b with $c=a\otimes b$.

We now turn to Theorem 3.2. We have already shown that

$$c=a\otimes b$$

for some $a\in \mathrm{GL}(U)$ and $b\in\mathrm{GL}(V)$; we must show that $a\in\mathrm{GL}(U,X)$ and $b\in\mathrm{GL}(V,Y)$. Choose $x\in X$ and $y\in Y$ so that $x\otimes y\in Z$. As $c\in\mathrm{GL}(W,Z)$, we have

$$(ax)\otimes(by)=c(x\otimes y)=x'\otimes y'$$

for some $x'\in X$ and $y'\in Y$. Hence for some $\lambda\in\mathbb{F}$ we have

$$ax=\lambda x',$$

$$by=\lambda^{-1}y'.$$

But X and Y are homogeneous; hence $ax\in X$ and $by\in Y$, as required.

The proof of Theorem 3.3 requires more care. We still have that $c=a\otimes b$, but a and b are not uniquely determined by this condition; they may be replaced by αa and $\alpha^{-1}b$. If α is chosen incorrectly, we shall not be able to conclude that $\alpha a \in \mathrm{GL}(U, X)$.

By the axiom of choice choose $x' = x'(x, y) \in X$ and $y' = y'(x, y) \in Y$ such that

$$(ax) \otimes (by) = c(x \otimes y) = x'(x, y) \otimes y'(x, y)$$

for $x \in X$, $y \in Y$. Conclude that there exists $\lambda = \lambda(x, y) \in \mathbb{F}$ such that

$$\begin{aligned} ax &= \lambda(x, y) x'(x, y), \\ by &= \lambda(x, y)^{-1} y'(x, y). \end{aligned} \tag{5.3}$$

Since X and Y are assumed to be circled, it is enough to prove that $|\lambda(x, y)| = 1$.

Fix $x_0 \in X$ and $y_0 \in Y$. Replacing a by $\lambda(x_0, y_0)^{-1}a$ and b by $\lambda(x_0, y_0)b$, we may assume w.l.o.g. that $ax_0 = x'(x_0, y_0)$ and $by_0 = y'(x_0, y_0)$, so that

$$\lambda(x_0, y_0) = 1.$$

Read y_0 for y in (5.3):

$$ax = \lambda(x, y_0) x'(x, y_0),$$

whence

$$\lambda(x, y) x'(x, y) = \lambda(x, y_0) x'(x, y_0).$$

As $x'(x, y) \in X$, $x'(x, y_0) \in X$, and X is antiradial, it follows that

$$|\lambda(x, y)| = |\lambda(x, y_0)|.$$

Similarly

$$|\lambda(x, y)| = |\lambda(x_0, y)|.$$

Hence

$$\begin{aligned} |\lambda(x, y)| &= |\lambda(x_0, y)| \\ &= |\lambda(x_0, y_0)| \\ &= 1, \end{aligned}$$

as required.

6. MORE NOTATION

Now assume $U, V, W_1, \ldots, W_n$ are normed. Define

$$B(V) = \{y \in V : \|y\| < 1\},$$

$$\bar{B}(V) = \{y \in V : \|y\| \leqslant 1\},$$

$$S(V) = \{y \in V : \|y\| = 1\},$$

$$E(V) = \text{extreme points of } \bar{B}(V),$$

$$\bar{E}(V) = \text{closure of } E(V),$$

so that by the Krein-Milman theorem (see e.g. [1]),

$$\bar{B}(V) = \text{convex hull of } \bar{E}(V).$$

We shall say that $S(V)$ is *smooth* at $x \in S(V)$ iff there is a unique support functional $\xi \in V^*$ to $\bar{B}(V)$ at x; i.e. for all $\eta \in V^*$ we have

$$\langle \eta, x \rangle = \|\eta\| = \|x\| = 1 \quad \Leftrightarrow \quad \eta = \xi.$$

We call $S(V)$ *smooth* iff it is smooth at each of its points. The reason for this terminology is the fact that $S(V)$ is smooth at x iff for all $v \in V$ the convex function $t \mapsto \|x + tv\|$ is differentiable at $t = 0$. (The proof of this fact is essentially the same as the proof of the Hahn-Banach theorem; see e.g. [1, pp. 445–453].)

We always give the space $L^n(W_1, \ldots, W_n; V)$ the *operator norm*:

$$\|A\| = \sup\left\{ \|A(x_1, \ldots, x_n)\| : x_i \in S(W_i) \right\}.$$

In the special case of the dual space U^* this norm is called the *dual norm*.

Denote by $I(U,V)$ the set of isometries from U onto V:

$$I(U,V) = \left\{ b \in L_{\text{iso}}(U,V) : \|b\| = \|b^{-1}\| = 1 \right\};$$

by $I(V)$ the isometry group of V:

$$I(V) = I(V,V);$$

and by $i(V)$ its Lie algebra:

$$i(V)=\{A\in \mathrm{gl}(V):\exp(tA)\in I(V)\ \forall t\in \mathbb{R}\}.$$

Note the obvious formulas

$$I(V)=\mathrm{GL}(V,Y), \tag{6.1}$$

$$i(V)=\mathrm{gl}(V,Y), \tag{6.2}$$

for $Y=B(V)$, $\bar{B}(V)$, $S(V)$, $E(V)$, $\bar{E}(V)$.

7. THE CANONICAL NORM ON W

The *canonical norm* on the tensor product $W=W_1\otimes\cdots\otimes W_n$ is characterized by each of the following three conditions:

$$\bar{B}(W)=\text{convex hull of } M_0, \tag{7.1}$$

where M_0 is the image of $S(W_1)\times\cdots\times S(W_n)$ by μ:

$$M_0=\{x_1\otimes\cdots\otimes x_n : x_j\in S(W_j)\};$$

(7.2) for every normed space V the canonical linear isomorphism

$$\mu^*: L(W,V)\to L^k(W_1,\dots,W_n;V)$$

[see (1.1)] is an isometry;

(7.3) the canonical isomorphism

$$W\simeq L^k(W_1,\dots,W_n;\mathbb{F})^*$$

is an isometry.

The equivalence of these three definitions is due to Grothendieck [3, p. 28]. Condition (7.2) explains the name "canonical": the tensor product W is often defined axiomatically by the requirement that the map μ^* be a linear isomorphism (see Section 1). For the convenience of the reader we sketch the proof of the equivalence of (7.1), (7.2), (7.3).

First define the canonical norm $\|\cdot\|$ to be the Minkowski functional of the right side of (7.1); then (7.1) is true by construction. One easily derives the formula

$$\|w\| = \inf\left\{ \sum_r \|x_{1r}\| \cdots \|x_{nr}\| \right\},$$

where the infimum is over all representations

$$w = \sum_r x_{1r} \otimes \cdots \otimes x_{nr}$$

of w as a sum of elements of M. For $A \in L(W,V)$ we have

$$\begin{aligned} \|Aw\| &= \left\| \sum_r (\mu^* A)(x_{1r}, \ldots, x_{nr}) \right\| \\ &\leqslant \|\mu^* A\| \sum_r \|x_{1r}\| \cdots \|x_{nr}\| \end{aligned}$$

so that taking the infimum gives $\|Aw\| \leqslant \|\mu^* A\| \, \|w\|$. Hence $\|A\| \leqslant \|\mu^* A\|$. On the other hand, if $\|x_1\| = \cdots \|x_n\| = 1$, then $x_1 \otimes \cdots \otimes x_n \in S(W)$, so that $\|x_1 \otimes \cdots \otimes x_n\| = 1$. Hence $\|\mu^*\| = 1$, so $\|\mu^* A\| = \|A\|$, proving (7.2). Assertion (7.3) is obtained by taking $V = \mathbb{F}$ in (7.2) and dualizing.

WARNING. The canonical linear isomorphism

$$W \simeq L^n(W_1^*, \ldots, W_n^*; \mathbb{F})$$

is not isometric. If the W_j are Hilbert spaces, this would imply that W and W^* are isometrically isomorphic, which is *not* the case.

THEOREM 7.4. *The set $E(W)$ of extreme points of $\bar{B}(W)$ is the image by μ of $E(W_1) \times \cdots \times E(W_n)$:*

$$E(W) = \{ x_1 \otimes \cdots \otimes x_n : x_j \in E(W_j) \}.$$

Proof. The map μ restricts to a local homeomorphism:

$$\mu : S(W_1) \times \cdots \times S(W_n) \to M_0 \subset S(W).$$

Hence if some $x_j \in S(W_j) \setminus E(W_j)$ (so that there is a line segment $L_j \subset S(W_j)$ with x_j in its interior), the point $x = x_1 \otimes \cdots \otimes x_n \in S(W) \setminus E(W)$ (as it lies in the interior of the line segment $L = x_1 \otimes \cdots \otimes x_{j-1} \otimes L_j \otimes x_{j+1} \otimes \cdots \otimes x_n$).

Conversely suppose $x_j \in E(W_j)$ $(j = 1, \ldots, n)$ but

$$x = x_1 \otimes \cdots \otimes x_n = \tfrac{1}{2}(u + v), \tag{7.5}$$

where $u, v \in \bar{B}(W)$. We must show that $x = u = v$. By the Krein-Milman theorem and the definition of $\bar{B}(W)$, u and v are convex combinations of elements of M_0. Thus let

$$u = \sum_r \alpha_r u_{1r} \otimes \cdots \otimes u_{nr},$$

$$v = \sum_s \beta_s v_{1s} \otimes \cdots \otimes v_{ns},$$

where $u_{jr}, v_{js} \in \bar{E}(W_j)$, $\alpha_r, \beta_s > 0$, and $\Sigma \alpha_r = \Sigma \beta_s = 1$.

Let ξ_j support $\bar{B}(W_j)$ at x_j; i.e.

$$\|\xi_j\| = \langle \xi_j, x_j \rangle = \|x_j\| = 1.$$

Let $\xi = \xi_1 \otimes \cdots \otimes \xi_n$. Then

$$1 = \langle \xi, x \rangle = \tfrac{1}{2}\langle \xi, u \rangle + \tfrac{1}{2}\langle \xi, v \rangle,$$

so

$$\langle \xi, u \rangle = \langle \xi, v \rangle = 1,$$

so

$$1 = \langle \xi, u \rangle = \sum_r \alpha_r \prod_j \langle \xi_j, u_{jr} \rangle$$

and

$$1 = \langle \xi, v \rangle = \sum_s \beta_s \prod_j \langle \xi_j, v_{js} \rangle;$$

so for all r, s,

$$1 = \prod_j \langle \xi_j, u_{jr} \rangle = \prod_j \langle \xi_j, v_{js} \rangle,$$

so (rescaling if necessary)

$$1 = \langle \xi_j, u_{jr} \rangle = \langle \xi_j, v_{js} \rangle$$

for all r, s and $j = 1, \dots, n$. Now apply $\xi_1 \otimes \cdots \otimes \xi_{j-1} \otimes \xi_{j+1} \otimes \cdots \otimes \xi_n$ to (7.5) to obtain

$$x_j = \tfrac{1}{2} \sum_r \alpha_r u_{jr} + \tfrac{1}{2} \sum_s \beta_s v_{js},$$

which (as x_j is extreme) is only possible if $x_j = u_{jr} = v_{js}$. Hence $x = u = v$, as required. ∎

8. ISOMETRIES OF W

Define $G_{ij} \subset L_{\text{iso}}(W_i, W_j)$ by

$$G_{ij} = I(W_i, W_j).$$

The system G_{ij} satisfies the axioms (2.1)–(2.3), so we may form the wreath tensor product $G \subset \mathrm{GL}(W)$. Endow W with the canonical norm. By Guiding Principle 2.5 we have

$$G \subset I(W). \tag{8.1}$$

We also assert

$$I(W) \cap \mathrm{GL}(W, M) \subset G. \tag{8.2}$$

Indeed, (8.2) holds more generally for any *cross norm* on W (see [2]), i.e. any norm which satisfies the equivalent conditions

$$\|x\| = \|x_1\| \cdots \|x_n\|$$

for $x = x_1 \otimes \cdots \otimes x_n$, and

$$M_0 \subset S(W).$$

This is because of the fact that for fixed $x_1 \in S(W_1), \dots,\ x_{j-1} \in S(W_{j-1})$,

$x_{j+1} \in S(W_{j+1}), \dots, x_n \in S(W_n)$, the map

$$W_j \to W : x_j \mapsto x_1 \otimes \cdots \otimes x_n$$

is an isometric embedding, and an element of $\mathrm{GL}(W, M)$ permutes the images (as the x_i and j vary) of these embeddings. Thus for $a \in I(W) \cap \mathrm{GL}(W, M)$ of form (2.4) we must have $a_j \in I(W_{\sigma(j)}, W_j) = G_{\sigma(j)j}$, whence $a \in G$ as required.

Combining (8.1) and (8.2) gives

$$I(W) \cap \mathrm{GL}(W, M) = G. \tag{8.3}$$

We would like to know when $I(W) = G$, i.e., when

$$I(W) \subset \mathrm{GL}(W, M). \tag{8.4}$$

Examples 4.3 and 4.4 show that we can't even conclude

$$i(W) \subset \mathrm{gl}(W, M) \tag{8.5}$$

without an additional hypothesis. Example 4.1 shows that (8.4) can fail even when (8.5) holds.

THEOREM 8.6. *If each set $E(W_j)$ is linearly prime, then*

$$i(W) = \mathcal{G}.$$

THEOREM 8.7. *If each $S(W_j^*)$ is smooth, then*

$$I(W) = G.$$

(By duality, the conclusion also follows if each $S(W_j)$ is smooth.)

To prove Theorem 8.6, take $Z_j = E(W_j)$ in Theorem 3.3, use Theorem 7.4 to conclude that $Z = E(W)$, and then apply (6.2). Before proving Theorem 8.7 we need some preliminary work.

According to (7.3), an element of $\xi \in W^*$ can be identified with a multilinear map $A: W_1 \times \cdots \times W_n \to \mathbb{F}$. As $(x_1, \dots, x_n)$ ranges over $S(W_1) \times \cdots \times S(W_n)$, the number

$$|\langle \xi, x \rangle| = |A(x_1, \dots, x_n)|$$

(where $x = x_1 \otimes \cdots \otimes x_n$) assumes its maximum (viz. $\|\xi\| = \|A\|$) at some nonempty set of n-tuples. One immediately concludes that $|\langle \xi, w\rangle|$ $(w \in W)$ assumes its maximum at any convex combination of elements $x \in M_0$ where $|\langle \xi, x\rangle|$ is maximal. Our next proposition asserts that such a convex combination is the most general such w.

Proposition 8.8. *Let $\xi \in S(W^*)$, and denote by F the set of all support functionals to $\bar{B}(W^*)$ at ξ:*

$$F = \{w \in S(W) : \langle \xi, w\rangle = 1\}.$$

Then

$$F = \textit{convex hull of } M_0 \cap F.$$

Proof. Choose $w \in F$. By (7.1)

$$w = \sum \beta_r x_r,$$

where $x_r \in M_0$, $\beta_r > 0$, $\Sigma \beta_r = 1$. As $w \in F$,

$$\langle \xi, w\rangle = \|\xi\| = \|w\| = 1.$$

Hence $\langle \xi, x_r\rangle = 1 = \|x_r\|$, so $x_r \in M_0 \cap F$, as required. (The reverse inclusion is even more obvious.) ■

Lemma 8.9. *Suppose $S(W^*)$ is smooth at $\eta \in S(W^*)$, and $y \in S(W)$ is the (unique) support functional at η. Then $y \in M_0$.*

Proof. This follows immediately from Proposition 8.8. ■

Lemma 8.10. *Suppose each $S(W_i^*)$ is smooth at $\xi_i \in S(W_i^*)$. Then $S(W^*)$ is smooth at $\xi = \xi_1 \otimes \cdots \otimes \xi_n$.*

Proof. Let $x_i \in S(W_i)$ be the unique support functional to ξ_i, and set $x = x_1 \otimes \cdots \otimes x_n$. Then x supports $\bar{B}(W)$ and ξ; we must show it is unique.

To this end suppose that $x' \in S(X)$ also supports $\bar{B}(W^*)$ at ξ; we must show $x = x'$. By Proposition 8.8 we may assume without loss of generality that $x' \in M_0$; then $x' = x_1' \otimes \cdots \otimes x_n'$, where $x_i' \in S(W_i)$. Then as

$$1 = \langle \xi, x'\rangle = \prod \langle \xi_i, x_i'\rangle$$

and

$$|\langle \xi_j, x_j' \rangle| \leq 1,$$

we have

$$\langle \xi_j, x_j' \rangle = \lambda_j,$$

where

$$1 = |\lambda_j| = \prod \lambda_j.$$

By smoothness at ξ_j,

$$\lambda_j^{-1} x_j' = x_j,$$

so $x' = x$, as required. ∎

Proof of Theorem 8.7. By (8.3) we need only show that $I(W) \subset GL(W, M)$. Hence choose $a \in I(W)$; we shall prove $a(M_0) = M_0$. Hence choose $x = x_1 \otimes \cdots \otimes x_n \in M_0$; we must show $ax \in M_0$. Let $\xi_j \in S(W_j^*)$ support $\bar{B}(W_j)$ at x_j. Then x supports $\bar{B}(W^*)$ at $\xi = \xi_1 \otimes \cdots \otimes \xi_n$, and by Lemma 8.10 x is the unique support functional to $\bar{B}(W^*)$ at ξ. Hence $y = ax$ is the unique support functional to $\bar{B}(W^*)$ at $\eta = a^{*-1}\xi$. Hence by Lemma 8.9, $y = ax \in M_0$, as required. ∎

9. REMARKS AND CONJECTURES

Using the results of Section 3, it is possible to remove the assumption that each Z_j is linearly prime in Theorems 3.1–3.3. Indeed, if each Z_j is not linearly prime, decompose Z_j to the union of linearly prime sets

$$Z_j = \bigcup_{k=1}^{\nu_j} Z_{jk}, \operatorname{span} Z_{jk} \cap \operatorname{span}\left(\bigcup_{\substack{1 \leq l \leq \nu_j \\ l \neq k}} Z_{jl} \right) = [0], \quad k = 1, \ldots, \nu_j. \tag{9.1}$$

Then

$$\begin{aligned} W_j &= \sum_{k=1}^{\nu_j} \oplus W_{jk}, \quad W_j = \operatorname{span} Z_j, \qquad W_{jk} = \operatorname{span} Z_{jk}, \\ W &= W_1 \otimes \cdots \otimes W_n = \sum_{\substack{1 \leq j_i \leq \nu_i \\ i = 1, \ldots, n}} \oplus W_{1j_1} \otimes \cdots \otimes W_{nj_n} \end{aligned} \tag{9.2}$$

Denote by $Z_{j_1\cdots j_n}$ and by $M_{j_1\cdots j_n}$ the images of $Z_{1j_1}\times\cdots\times Z_{nj_n}$ and $W_{1j_1}\times\cdots\times W_{nj_n}$ respectively under the canonical map $\mu\in L(W_{1j_1},\ldots,W_{nj_n};W_{1j_1}\otimes\cdots\otimes W_{nj_n})$. According to Theorem 3.1 each $Z_{j_1\cdots j_n}$ is linearly prime. Then we have

THEOREM 9.3. *Let* (9.1) *the unique decomposition of the set* Z_j *to linearly prime factors. Then*

$$\operatorname{gl}(W,Z)\subset\sum_{\substack{1\leqslant j_i\leqslant \nu_i\\ i=1,\ldots,n}}\oplus\operatorname{gl}\left(W_{1j_1}\otimes\cdots\otimes W_{nj_n},M_{j_1\cdots j_n}\right). \tag{9.4}$$

If in addition each Z_{jk}, $1\leqslant j\leqslant n$, $1\leqslant k\leqslant \nu_j$, *is homogeneous* (*circled and antiradial*), *then equality holds in* (9.4).

Let V be normed, and suppose that $E(V)=Y=\cup_{k=1}^{\nu}Y_k$ is the decomposition of Y to linearly prime factors. Put $V_k=\operatorname{span}Y_k$. Denote by $\|\ \|_k$ the restriction of the norm $\|\ \|$ on V to V_k. We claim that the norm on V has a simple representation by the norm $\|\ \|_k$.

LEMMA 9.5. *Let the above assumption hold. Put*

$$v=\sum_{i=1}^{\nu}v_i,\quad v_i\in V_i,\qquad i=1,\ldots,\nu.$$

Then

$$\|v\|=\sum_{i=1}^{\nu}\|v_i\|_i. \tag{9.6}$$

Proof. Define a new norm on V:

$$|||v|||=\sum_{i=1}^{\nu}\|v_i\|_i.$$

Our lemma will follow if the set of the extreme points $E'(V)$ of the norm $\|\cdot\|$ coincides with the set Y. Suppose that $|||v|||=1$, and assume for simplicity that each $v_i\neq 0$. Then

$$v=\sum_{i=1}^{\nu}\|v_i\|_iu_i,\qquad u_i=\frac{v_i}{\|v_i\|}.$$

So v is a nontrivial convex combination of $u_1, \ldots, u_\nu$, where each u_i is in $S(V)$ and V_i. Thus

$$E'(V) = \bigcup_{i=1} E(V_i).$$

Clearly $Y_i \subset E(V_i)$. It is left to show that $Y_i \supset E(V_i)$. The classical result claims that any $v \in B(V)$ is a finite convex combination of vectors from $E(V)$. So for any $v \in B(V)$

$$v = \sum_{i=1}^{\nu} \sum_{y_{ij} \in Y_i} \alpha_{ij} y_{ij}, \qquad \alpha_{ij} > 0, \quad \sum_{i,j} \alpha_{ij} = 1.$$

As

$$V = V_1 \oplus \cdots \oplus V_\nu, \qquad V_i = \operatorname{span} Y_i,$$

the assumption that $v \in V_i$ implies

$$v = \sum_{y_{ij} \in Y_i} \alpha_{ij} y_{ij}, \qquad \alpha_{ij} > 0, \quad \sum_j \alpha_{ij} = 1.$$

Thus $Y_i = E(V_i)$ and $|||v||| = \|v\|$, as required. ∎

Next we focus our attention on $\mathrm{GL}(W, Z)$, where W has the decomposition (9.2). Clearly a given $a \in \mathrm{GL}(W, Z)$ may shift between the isomorphic components $W_{1j_1} \otimes \cdots \otimes W_{nj_n}$. Thus to analyze $\mathrm{GL}(W, Z)$ completely it suffices to study the case where each Z_j is linearly prime. Theorems 3.2–3.3 determined completely the structure of the Lie algebra of $\mathrm{GL}(W, Z)$. Example 4.1 gave us the cases in which $\mathrm{GL}(W, Z)$ was bigger than G.

We conclude our paper with the following conjecture.

Conjecture 9.7. Let the assumptions of Theorem 3.2 (3.3) hold. Then (up to isomorphism) we can decompose each W_j as a tensor product $W_{j1} \otimes W_{j2} \otimes \cdots \otimes W_{j\nu_j}$ such that each Z_j is the image of $Z_{j1} \times \cdots \times Z_{j\nu_j}$ under the canonical map μ, where each Z_{jk} is linearly prime and homogeneous (circled and antiradial). Moreover $\mathrm{GL}(W, Z)$ coincides with G' generated by the factors

$$G_{(i,p)(j,q)} = \left\{ a \in L_{\mathrm{iso}}\left(W_{ip}, W_{jq}\right), \, a\left(Z_{ip}\right) = Z_{jq} \right\}.$$

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